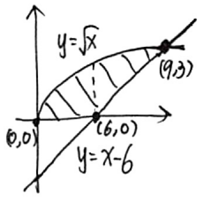
Week 14 Worksheet Thursday - Area & Volume computing

Instructions. Follow the instructions given by your TA. You are not expected to finish all the problems. :)

1. Find the area in 1st quadrant that is bounded above by $y = \sqrt{x}$ and below by x axis and $y = x - 6$.



Intersection point : $(9,3)$

$\sqrt{x} = x - 6$

$(\sqrt{x}-3)(\sqrt{x}+2)=0$

$\Rightarrow x=9$

Compute area (2 approaches)

$\boxed{\text{Approach 1 } \int \cdots dx}$

$$\int_0^6 \sqrt{x}\,dx + \int_6^9 \sqrt{x} - x + 6\,dx$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Big|_{x=0}^{x=6} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Big|_{x=6}^{x=9} - \frac{x^2}{2}\Big|_{x=6}^{x=9} + 6x\Big|_{x=6}^{x=9}$$

$$= \frac{2}{3}\cdot 9^{\frac{3}{2}} - \frac{81}{2} + \frac{36}{2} + 54 - 36$$

$$= 18 - 40.5 + 18 + 18 = 54 - 40.5 = 13.5$$

$\boxed{\text{Approach 2 } \int \cdots dy}$ (easier!)

$y=\sqrt{x} \Rightarrow x = y^2$

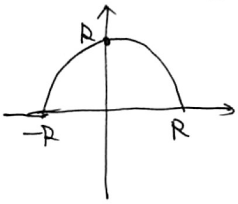
$y = x-6 \Rightarrow x = y+6$

$$\int_0^3 y+6-y^2\,dy$$

$$= \frac{y^2}{2} + 6y - \frac{y^3}{3}\Big|_{y=0}^{y=3}$$

$$= \frac{9}{2} + 18 - 9 = 4.5 + 9 = 13.5$$

2. The region underneath $y = \sqrt{R^2 - x^2}$ is revolved around the x axis. Compute the volume. (R is some constant.)



$$\int_{-R}^{R} \pi(R^2 - x^2)\,dx$$

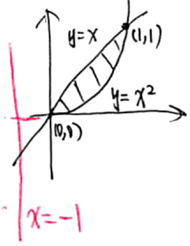
$$= \int_{-R}^{R} \pi R^2 - \pi x^2\,dx$$

$$= \pi R^2 x\Big|_{x=-R}^{x=R} - \pi\frac{x^3}{3}\Big|_{x=-R}^{x=R}$$

$$= 2\pi R^3 - \frac{2}{3}\pi R^3 = \frac{4}{3}\pi R^3$$

This is how the formula of volume for sphere is derived.

3. Consider the region bounded by $y = x$ and $y = x^2$, $0 \le x \le 1$.
   (a) Compute its area
   (b) Rotate this region around x axis. What's the volume?
   hard (optional) $\boxed{\text{(c)}}$ Rotate this region aound the line $x = -1$. What is the volume?



(a)
$$\int_0^1 x - x^2\,dx$$

$$= \frac{x^2}{2} - \frac{x^3}{3}\Big|_{x=0}^{x=1}$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

(b) $V = V_{outer} - V_{hole}$

$$= \int_0^1 \pi x^2\,dx - \int_0^1 \pi (x^2)^2\,dx$$

$$= \pi\frac{x^3}{3}\Big|_{x=0}^{x=1} - \pi\frac{x^5}{5}\Big|_{x=0}^{x=1}$$

$$= \frac{\pi}{3} - \frac{\pi}{5} = \frac{2}{15}\pi$$

(c) [optional]

$y = x \Rightarrow x = y$ (inner curve)

$y = x^2 \Rightarrow x = \sqrt{y}$ (outer curve)

$V = V_{outer} - V_{hole}$

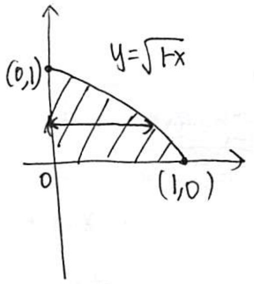
$$= \int_0^1 \pi(1+\sqrt{y})^2\,dy - \int_0^1 \pi(1+y)^2\,dy$$

$$= \pi\int_0^1 \cancel{1} + 2y^{\frac{1}{2}} + y \cancel{-1} - y^2 - 2y\,dy$$

$$= \pi\int_0^1 2y^{\frac{1}{2}} - y - y^2\,dy$$

$$= \pi\left(2\frac{y^{\frac{3}{2}}}{\frac{3}{2}} - \frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_{y=0}^{y=1} = \pi\left(\frac{4}{3} - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{2}\pi.$$

4. R is the region below the curve $y = \sqrt{1-x}$ between $x = 0$, $x = 1$, and above the x-axis. Compute the volume of the solid obtained by rotation of the region R around y-axis.



$y = \sqrt{1-x} \Rightarrow y^2 = 1-x$

$x = 1-y^2$

$$\int_0^1 \pi (1-y^2)^2 \, dy$$

$$= \int_0^1 \pi (1-2y^2+y^4) \, dy$$

$$= \pi \left(y - 2\frac{y^3}{3} + \frac{y^5}{5} \right) \Big|_{y=0}^{y=1}$$

$$= \pi \left(1 - \frac{2}{3} + \frac{1}{5} \right)$$

$$= \frac{8}{15} \pi$$